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Two-variable Hermite Polynomial State and Its Wigner Function

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Abstract In this paper we obtain the Wigner functions of two-variable Hermite polynomial states (THPS) and their marginal distribution using the entangled state $|\xi\rangle$ representation. Also we obtain tomogram of THPS by virtue of the Radon transformation between the Wigner operator and the projection operator of another entangled state $|\eta, \tau_1, \tau_2\rangle$.

Keywords Two-variable Hermite polynomial state · Entangled state representation · Wigner function · Tomogram function

1 Introduction

Squeezed states [1] has been a major topic in quantum optics since 1970s. The quantum states of squeezed light can be measured by optical tomography which is based on the relation between the marginal distribution function for photon homodyne quadrature and the Wigner function [2, 3]. In theory, Bergou et al. [4] found that the single-variable Hermite polynomial state (SHPS) is the minimum uncertainty state for the variables Y_1 and Y_2 which describe amplitude-squared squeezing, where the operators $Y_1 = [a^2 + a^{\dagger 2}]/2$ and $Y_2 = [a^2 - a^{\dagger 2}]/(2\mathrm{i})$ obey the uncertainty relation $\Delta Y_1 \Delta Y_2 \geq \langle N + 1/2\rangle$. The minimum uncertainty state, which makes the uncertainty relation minimum $\Delta Y_1 \Delta Y_2 = \langle N + 1/2\rangle$, is taken the following form

$$|m, \nu\rangle = C_m(\nu) S(\xi) H_m(\mathrm{i}\gamma(\nu) a^\dagger)|0\rangle, \tag{1}$$

where $H_m(x)$ is the m-order Hermite polynomial [5],

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$$H_m(x)=\sum_{k=0}^{[m/2]}\frac{(-)^k m!}{k!(m-2k)!}(2x)^{m-2k} \tag{2}$$

and $C_m(\nu)$ is the normalization constant,

$$[C_m(\nu)]^{-2}=\sum_{n=0}^{[m/2]}\frac{(m!)^2[4|\gamma(\nu)|^2]^{m-2n}}{(m-2n)!(n!)^2}. \tag{3}$$

The SHPS is of physical interest because it can be generated by a degenerate parametric amplifier [4]. Especially, for the state $|1,\nu\rangle$ it can be produced by using a one-photon number state as the input for a degenerate parametric amplifier because it is a squeezed one-photon state. Recently, Fan et al. [6] first introduced the concept of Sum-Frequency squeezing for the variables Z_1 and Z_2,

$$Z_1=\frac{1}{2}(a_1^\dagger a_2^\dagger+a_1a_2),\qquad Z_2=\frac{\mathrm{i}}{2}(a_1^\dagger a_2^\dagger-a_1a_2), \tag{4}$$

which obey the uncertainty relation $\Delta Z_1\Delta Z_2\geq\frac{1}{4}\langle N_1+N_2+1\rangle$. If $(\Delta Z_j)^2<\frac{1}{4}\langle N_1+N_2+1\rangle$, the state is said to be Sum-Frequency squeezing in the Z_j direction. The minimum uncertainty states that make the uncertainty relation minimum $\Delta Z_1\Delta Z_2=\frac{1}{4}\langle N_1+N_2+1\rangle$ for Sum-Frequency squeezing are the solutions to the eigenvalue equation:

$$(Z_1+\mathrm{i}\chi Z_2)|\psi\rangle=\beta|\psi\rangle, \tag{5}$$

where $\beta=\frac{1}{2}\sqrt{\chi^2-1}(M+1)$, M is nonnegative integer and $\chi\geqslant 1$. Through concise calculation, the state $|\psi\rangle$ is expressed as,

$$|\psi\rangle=C_{m,n}(\chi)S_2(r)H_{m,n}(fa_1^\dagger,fa_2^\dagger)|00\rangle, \tag{6}$$

where $f^2=-\sqrt{\chi^2-1}/\chi$, $S_2(r)=\exp[r(a_1^\dagger a_2^\dagger-a_1a_2)]$ is the two-mode squeezed operator and the normalization constant is

$$C_{m,n}(\chi)=\left[\sum_{l=0}^{\min(m,n)}\binom{m}{l}\binom{n}{l}m!n!|f|^{2(m+n-2l)}\right]^{1/2}. \tag{7}$$

Because $H_{m,n}(\epsilon,\epsilon^*)$ is the two-variable Hermite polynomial [5],

$$H_{m,n}(\epsilon,\epsilon^*)=\sum_{l=0}^{\min(n,m)}\frac{(-)^l n!m!}{l!(m-l)!(n-l)!}\epsilon^{m-l}\epsilon^{*n-l}, \tag{8}$$

the state $|\psi\rangle$ is named as the two-variable Hermite polynomial state (THPS). Moreover, Fan and Ye [6] emphasized that amplitude-squared squeezing is the limit of Sum-Frequency squeezing because the SHPS is the limit of the THPS in degenerate conditions.

The aim of this paper is to find a concise method for obtaining Wigner functions of the THPS. To achieve this aim, it would be convenient to use the Wigner operator in the entangled state $|\xi\rangle$ representations [7]. The paper is organized as follows: In Sect. 2 we recall the bipartite entangled state $|\xi\rangle$ representation and its properties. In Sect. 3 we find the relation between the THPS and the squeezed number state (SNS). In Sect. 4 we obtain Wigner function of THPS and its two marginal distributions using the entangled Wigner operator. In Sect. 5 we obtain tomogram of THPS by virtue of the Radon transformation between the Wigner operator and the projection operator of another entangled state $|\eta,\tau_1,\tau_2\rangle$ [7].

2 Bipartite Entangled State Representation

Let Q be the two particles' center-of-mass coordinate $Q=Q_1+Q_2$ and P_r be the relative momentum $P_r=P_1-P_2$, where Q_i, P_i are related to boson operators by $Q_i=(a_i+a_i^\dagger)/\sqrt{2}$, $P_i=(a_i-a_i^\dagger)/(\sqrt{2}\mathrm{i})$, $[a_i,a_j^\dagger]=\delta_{ij}$, $i,j=1,2$. Due to the fact that operators Q and P_r are commutative, i.e., $[Q,P_r]=0$, they have the common eigenvector $|\xi\rangle$, which is expressed as [8]

$$|\xi\rangle=\exp\left[-\frac{1}{2}|\xi|^2+\xi a_1^\dagger+\xi^* a_2^\dagger-a_1^\dagger a_2^\dagger\right]|00\rangle, \tag{9}$$

where $|00\rangle$ is two-mode vacuum state and ξ is complex, $\xi=\xi_1+\mathrm{i}\xi_2$. It can be proved that

$$Q|\xi\rangle=\sqrt{2}\xi_1|\xi\rangle, \qquad P_r|\xi\rangle=\sqrt{2}\xi_2|\xi\rangle. \tag{10}$$

Using the technique of integration within an ordered product (IWOP) of operators [9], we can prove the orthogonal property and completeness relation

$$\langle\xi|\xi'\rangle=\pi\delta(\xi_1-\xi_1')\delta(\xi_2-\xi_2'), \qquad \int\frac{d^2\xi}{\pi}|\xi\rangle\langle\xi|=1, \qquad d^2\xi=d\xi_1 d\xi_2. \tag{11}$$

In $|\xi\rangle$ representation the squeezed operator $S_2(r)$ can be expressed as

$$S_2(r)=\mu\int\frac{d^2\xi}{\pi}|\mu\xi\rangle\langle\xi|, \quad \mu=e^r. \tag{12}$$

Operating the operator $S_2(r)$ on the state $|\xi\rangle$ leads to

$$S_2(r)|\xi\rangle=\mu|\mu\xi\rangle \tag{13}$$

and $S_2(r)$ leads to the Bogoliudov transforms, i.e.,

$$S_2^{-1}a_1^\dagger S_2=a_1^\dagger\cosh r+a_2\sinh r, \qquad S_2^{-1}a_2^\dagger S_2=a_2^\dagger\cosh r+a_1\sinh r, \tag{14}$$

$$S_2 a_1^\dagger S_2^{-1}=a_1^\dagger\cosh r-a_2\sinh r, \qquad S_2 a_2^\dagger S_2^{-1}=a_2^\dagger\cosh r-a_1\sinh r. \tag{15}$$

3 Relation Between the THPS and the SNS

In order to obtain the Wigner function of the THPS, we first find the relation between the THPS and the SNS. Owing to the generating function formula of $H_{m,n}(\epsilon,\epsilon^*)$ [10]

$$\sum_{m,n=0}^{\infty}\frac{t^m t'^n}{m!n!}H_{m,n}(\epsilon,\epsilon^*)=\exp(-tt'+t\epsilon+t'\epsilon^*), \tag{16}$$

where $H_{m,n}(\epsilon,\epsilon^*)$ possess the property

$$H^*_{m,n}(\epsilon,\epsilon^*) = H_{n,m}(\epsilon,\epsilon^*), \tag{17}$$

in two-mode Fock space the state $\langle\xi|$ is expanded as

$$\langle\xi| = \langle j,k|e^{-|\xi|^2/2}\sum_{j,k=0}^{\infty}\frac{1}{\sqrt{j!k!}}H_{k,j}(\xi,\xi^*), \tag{18}$$

where $\langle j,k| = \langle 00|\frac{a_1^j a_2^k}{\sqrt{j!k!}}$ is the two-mode number state. Thus the inner product between $\langle\xi|$ and $|q,p\rangle$ is

$$\langle\xi|q,p\rangle = e^{-|\xi|^2/2}\frac{1}{\sqrt{p!q!}}H_{p,q}(\xi,\xi^*). \tag{19}$$

Using (12)–(13) and (16) we obtain

$$\begin{aligned}
&\frac{\sqrt{p!q!}}{\sec hr}(\tanh r)^{-(p+q)/2}S_2(r)|q,p\rangle \\
&\quad= \frac{\mu(\tanh r)^{-(p+q)/2}}{\sec hr}\int\frac{d^2\xi}{\pi}|\mu\xi\rangle e^{-|\xi|^2/2}H_{p,q}(\xi,\xi^*) \\
&\quad= \frac{\mu(\tanh r)^{-(p+q)/2}}{\sec hr}\frac{\partial^{q+p}}{\partial t^p\partial t'^q}\int\frac{d^2\xi}{\pi}\exp\left\{-\frac{(\mu^2+1)|\xi|^2}{2}+(\mu a_1^\dagger+t)\xi\right. \\
&\qquad\left.+(\mu a_2^\dagger+t')\xi^*-tt'-a_1^\dagger a_2^\dagger\right\}|00\rangle|_{t=t'=0} \\
&\quad= (\tanh r)^{-(p+q)/2}\frac{\partial^{q+p}}{\partial t^p\partial t'^q}\exp\{-tt'\tanh r+t'a_1^\dagger\sec hr \\
&\qquad+ta_2^\dagger\sec hr+a_1^\dagger a_2^\dagger\tanh r\}||00\rangle_{t=t'=0} \\
&\quad= H_{q,p}\left(\frac{\sqrt{2}a_1^\dagger}{\sqrt{\sinh 2r}},\frac{\sqrt{2}a_2^\dagger}{\sqrt{\sinh 2r}}\right)\exp(a_1^\dagger a_2^\dagger\tanh r)|00\rangle.
\end{aligned} \tag{20}$$

Using the Bogoliubov transforms in (14)–(15), (20) can be converted into

$$\frac{\sqrt{p!q!}}{\sec hr}(\tanh r)^{-(p+q)/2}S_2(r)|q,p\rangle = S_2(r)H_{q,p}\left(\frac{a_1^\dagger}{\sqrt{\tanh r}},\frac{a_2^\dagger}{\sqrt{\tanh r}}\right)|00\rangle, \tag{21}$$

which show that the THPS may be regarded as the SNS. Comparison between (6) and (21) show that the squeezed parameter r must be smaller than or equal to zero. For the convenience of deriving the Wigner function and tomogram function of the THPS, we define $|\phi\rangle$ as

$$|\phi\rangle = C_{q,p}(r)\frac{\sqrt{p!q!}}{\sec hr}(\tanh r)^{-(p+q)/2}S_2(r)|q,p\rangle, \tag{22}$$

where $C_{q,p}(r)$ is the same of (7) while for $f = 1/\sqrt{\tanh r}$. From (6) and (21)–(22) we see that the state $|\phi\rangle$ is the normalized THPS in fact, so we may use the SNS $|\phi\rangle$ to discuss the THPS and its some properties.

4 Wigner Function of the THPS and Its Marginal Distributions

The two-mode Wigner operator $\Delta_1(x_1, p_1)\Delta_2(x_2, p_2)$ in $|\xi\rangle$ representation is expressed as [7]:

$$\Delta(\gamma,\rho) = \int \frac{d^2\xi}{\pi^3} |\gamma - \xi\rangle\langle\gamma + \xi| \exp(\xi^*\rho - \xi\rho^*), \tag{23}$$

where $\gamma = \alpha + \beta^*$, $\rho = \alpha - \beta^*$, $\alpha = (x_1 + \mathrm{i}p_1)/\sqrt{2}$, $\beta = (x_2 + \mathrm{i}p_2)/\sqrt{2}$. From (22) and (23), we obtain the Wigner function of the THPS

$$W(\rho,\gamma) = \langle\phi|\Delta(\rho,\gamma)|\phi\rangle = \frac{|C_{q,p}(r)|^2 p!q!}{\sec h^2 r (\tanh r)^{p+q}} \langle q, p|S_2^{-1}(r)\Delta(\rho,\gamma)S_2(r)|q, p\rangle. \tag{24}$$

Using (13) and (19) we obtain the inner products,

$$\langle\gamma + \xi|S_2(r)|q, p\rangle = \frac{1}{\mu\sqrt{p!q!}} H_{p,q}[(\gamma + \xi)/\mu, (\gamma + \xi)^*/\mu] \exp[-|\gamma + \xi|^2/2\mu^2], \tag{25}$$

$$\langle q, p|S_2^{-1}(r)|\gamma + \xi\rangle = \frac{1}{\mu\sqrt{p!q!}} H_{q,p}[(\gamma - \xi)/\mu, (\gamma - \xi)^*/\mu] \exp[-|\gamma - \xi|^2/2\mu^2], \tag{26}$$

so the inner product $\langle q, p|S_2^{-1}(r)\Delta(\rho,\gamma)S_2(r)|q, p\rangle$ is

$$\begin{aligned}
&\langle q, p|S_2^{-1}(r)\Delta(\rho,\gamma)S_2(r)|q, p\rangle \\
&= \langle q, p| \int \frac{d^2\xi}{\mu^2\pi^3} |(\gamma - \xi)/\mu\rangle\langle(\gamma + \xi)/\mu| \exp(\xi^*\rho - \xi\rho^*)|q, p\rangle \\
&= \frac{1}{\mu^2 p!q!} \int \frac{d^2\xi}{\pi^3} \exp[-|(\gamma - \xi)|^2/2\mu^2 - |(\gamma + \xi)|^2/2\mu^2 + \xi^*\rho - \xi\rho^*] \\
&\quad \times H_{p,q}[(\gamma + \xi)/\mu, (\gamma + \xi)^*/\mu] H_{q,p}[(\gamma - \xi)/\mu, (\gamma - \xi)^*/\mu] \\
&= \frac{1}{\mu^2\pi^2 p!q!} \frac{\partial^{q+p}}{\partial t^p \partial t'^q} \frac{\partial^{q+p}}{\partial s^q \partial s'^p} \int \frac{d^2\xi}{\pi^3} \exp\Big[-|\xi|^2/\mu^2 - |\gamma|^2/\mu^2 + \xi^*\rho - \xi\rho^* \\
&\quad - tt' + \frac{\gamma + \xi}{\mu} t + \frac{(\gamma + \xi)^*}{\mu} t' - ss' + \frac{\gamma - \xi}{\mu} s + \frac{(\gamma - \xi)^*}{\mu} s'\Big] \\
&= \frac{1}{\mu^2\pi^2 p!q!} \frac{\partial^{q+p}}{\partial t^p \partial t'^q} \frac{\partial^{q+p}}{\partial s^q \partial s'^p} \exp\left[-\frac{|\gamma|^2}{\mu^2} - tt' - ss' + \frac{t\gamma}{\mu} - \frac{s\xi}{\mu} + \frac{t'\gamma^*}{\mu} - \frac{s'\xi^*}{\mu}\right] \\
&\quad \times \int \frac{d^2\xi}{\pi^3} \exp\left[-\frac{|\xi|^2}{\mu^2} + \xi^*(\rho + \frac{t'}{\mu} - \frac{s'}{\mu}) + \xi(-\rho^* + \frac{t}{\mu} - \frac{s}{\mu})\right].
\end{aligned} \tag{27}$$

Further, using

$$\int \frac{d^2z}{\pi} \exp(\zeta|z|^2 + \xi z + \eta z^*) = -\frac{1}{\zeta} \exp\left(-\frac{\xi\eta}{\zeta}\right), \quad \mathrm{Re}\,\zeta < 0, \tag{28}$$

and (24) we obtain the Wigner function of the THPS

$$W(\rho,\gamma)=\frac{|C_{q,p}(r)|^2 e^{-(|\gamma|^2/\mu^2+\mu^2|\rho|^2)}}{\pi^2\sec hr(\tanh r)^{p+q}}H_{p,p}[(\gamma/\mu+\mu\rho),(\gamma/\mu+\mu\rho)^*]$$
$$\times H_{q,q}[(\gamma/\mu-\mu\rho),(\gamma/\mu-\mu\rho)^*]. \tag{29}$$

From (29) we see that the Wigner function $W(\rho,\gamma)$ gives rise to the quadrature squeezing, which are associated with the variances of the quadrature fluctuations. By reason that μ plays the role of a squeezing parameter, when $\mu<1$, we clearly see that the Wigner distribution can be compressed along the ρ direction at the expanse of an increase along the γ direction.

Noting the relationship between $H_{m,m}$ and the Laguerre polynomial L_m,

$$H_{m,m}(\varepsilon,\varepsilon^*)=m!(-1)^m L_m(|\varepsilon|^2), \tag{30}$$

we can put further (29) into

$$W(\rho,\gamma)=\frac{|C_{q,p}(r)|^2 e^{-(|\gamma|^2/\mu^2+\mu^2|\rho|^2)}}{\pi^2\sec hr(\tanh r)^{p+q}}p!q!(-1)^{p+q}L_p(|\gamma/\mu+\mu\rho|^2)L_q(|\gamma/\mu-\mu\rho|^2). \tag{31}$$

From (23) carrying out the integral over $d^2\rho$ for $\Delta(\rho,\gamma)$ leads to [7]

$$\int d^2\rho\,\Delta(\rho,\gamma)=\frac{1}{\pi}|\xi\rangle\langle\xi|_{\xi=\gamma}, \tag{32}$$

owing to

$$\langle\xi|S_2(r)|q,p\rangle=\frac{1}{\mu\sqrt{p!q!}}H_{p,q}(\xi/\mu,\xi^*/\mu)\exp(-|\xi|^2/2\mu^2), \tag{33}$$

so we obtain a marginal distribution of the Wigner function of the THPS in the γ variable

$$\int d^2\rho\,W(\rho,\gamma)=\frac{1}{\pi}|\langle\xi|\phi\rangle|^2_{\eta=\gamma}=\frac{|C_{p,q}(r)|^2}{\pi\mu^2\sec h^2r(\tanh r)^{p+q}}|H_{p,q}(\gamma/\mu,\gamma^*/\mu)|^2$$
$$\times\exp(-|\gamma|^2/\mu^2). \tag{34}$$

Similarly, performing the integration of $\Delta(\rho,\gamma)$ over $d^2\gamma$ leads to another projection operator, i.e.,

$$\int d^2\gamma\,\Delta(\rho,\gamma)=\frac{1}{\pi}|\eta\rangle\langle\eta|_{\eta=\rho}, \tag{35}$$

where the state $|\eta\rangle$ is the common eigenstate of two particles' relative position (Q_1-Q_2) and the total momentum (P_1+P_2) in two-mode Fock space [10], i.e.,

$$(Q_1-Q_2)|\eta\rangle=\sqrt{2}\eta_1|\eta\rangle,\qquad (P_1+P_2)|\eta\rangle=\sqrt{2}\eta_2|\eta\rangle, \tag{36}$$

its explicit expression is

$$|\eta\rangle=\exp[-|\eta|^2/2+\eta a_1^\dagger-\eta^* a_2^\dagger+a_2^\dagger a_1^\dagger]|00\rangle,\quad \eta=\eta_1+\mathrm{i}\eta_2, \tag{37}$$

which is capable of making up the complete entangled state representation. In this representation the squeezed operator $S_2(r)$ can be expressed as

$$S_2(r) = \int \frac{d^2\eta}{\mu\pi} |\eta/\mu\rangle\langle\eta|. \tag{38}$$

Using the expansion of $|\eta\rangle$ in the Fock space

$$|\eta\rangle = e^{-|\eta|^2/2} \sum_{j,k=0}^{\infty} \frac{(-1)^k}{\sqrt{j!k!}} H_{j,k}(\eta,\eta^*)|j,k\rangle \tag{39}$$

and (38) we obtain another marginal distribution of the Wigner function of the THPS in the ρ variable,

$$\int d^2\gamma\, W(\rho,\gamma) = \frac{\mu^2|C_{p,q}(r)|^2}{\pi \sec h^2 r(\tanh r)^{p+q}} |H_{p,q}(\mu\rho,\mu\rho^*)|^2 \exp(-\mu^2|\rho|^2). \tag{40}$$

Equations (34) and (40) are proportional to the probability for finding the two particles under the THPS in an entangled way in the $\rho-\gamma$ phase space.

5 Tomogram of the THPS

The use of tomograms in quantum mechanics and quantum optics provides the possibility of describing a quantum state with a positive probability distribution. A direct description of quantum states by means of quantum tomograms for the system observables is interesting from both the theoretical and experimental points of view. Therefore, in recent years tomogram approach has brought much interest of physicists. In this section we continue to derive the tomogram of the state $|\phi\rangle$, which is defined as [7]

$$T(\eta,\tau_1,\tau_2) = \pi \iint_{-\infty}^{\infty} d^2\rho d^2\gamma\, \delta(\eta_1 - \mu_1\gamma_1 - \nu_1\rho_2)\delta(\eta_2 - \nu_2\gamma_2 - \mu_2\rho_1) W(\rho,\gamma), \tag{41}$$

where τ_1, τ_2, ρ and γ are complex numbers, $\tau_j = |\tau_j|e^{i\theta} = \mu_j + i\nu_j$ $(j = 1, 2)$, $\rho = \rho_1 + i\rho_2$ and $\gamma = \gamma_1 + i\gamma_2$. However, it will be very tough to evaluate the integration if we directly substitute (29) into (41). Fortunately, we can use the following relation between the Wigner operator and the projection operator of the state $|\eta,\tau_1,\tau_2\rangle$

$$|\eta,\tau_1,\tau_2\rangle\langle\eta,\tau_1,\tau_2| = \pi \iint_{-\infty}^{\infty} d^2\rho d^2\gamma\, \delta(\eta_1 - \mu_1\gamma_1 - \nu_1\rho_2)\delta(\eta_2 - \nu_2\gamma_2 - \mu_2\rho_1)\Delta(\rho,\gamma). \tag{42}$$

Thus the tomogram of the state $|\phi\rangle$ is

$$\begin{aligned} T(\eta,\tau_1,\tau_2) &= \pi \iint_{-\infty}^{\infty} d^2\rho d^2\gamma\, \delta(\eta_1 - \mu_1\gamma_1 - \nu_1\rho_2)\delta(\eta_2 - \nu_2\gamma_2 - \mu_2\rho_1)\langle\phi|\Delta(\rho,\gamma)|\phi\rangle \\ &= |\langle\eta,\tau_1,\tau_2|\phi\rangle|^2, \end{aligned} \tag{43}$$

which shows that for tomographic approach there exists the entangled state $|\eta,\tau_1,\tau_2\rangle$, and the Radon transforms of the Wigner operator are just the entangled-state density matrices $|\eta,\tau_1,\tau_2\rangle\langle\eta,\tau_1,\tau_2|$. As a result, the tomogram of quantum states can be considered as the module-square of the states' wave function in this entangled state representation. This is a new way to derive tomograms of quantum states. Here the entangled state $|\eta,\tau_1,\tau_2\rangle$ is expressed as

$$|\eta,\tau_1,\tau_2\rangle = A\exp[B + Ca_1^\dagger + Da_2^\dagger + Ea_1^\dagger a_2^\dagger - Fa_1^{\dagger 2} - Fa_2^{\dagger 2}|00\rangle, \tag{44}$$

where

$$\begin{aligned} A &= \frac{1}{\sqrt{|\tau_1\tau_2|}}, \qquad B = -\frac{\eta_1^2}{2|\tau_1|^2} - \frac{\eta_2^2}{2|\tau_2|^2}, \qquad C = \frac{\eta_1}{\tau_1^*} + \frac{\eta_2}{\tau_2^*}, \\ D &= -\frac{\eta_1}{\tau_1^*} + \frac{\eta_2}{\tau_2^*}, \qquad E = \frac{1}{2}(e^{\mathrm{i}2\theta_1} - e^{\mathrm{i}2\theta_2}), \qquad F = \frac{1}{4}(e^{\mathrm{i}2\theta_1} + e^{\mathrm{i}2\theta_2}), \end{aligned} \tag{45}$$

then the tomogram amplitude of the state $|\phi\rangle$ is

$$\langle\phi|\eta,\tau_1,\tau_2\rangle = \frac{C_{q,p}(r)(\cosh r)^{p+q+1}}{(\tanh r)^{(p+q)/2}}\langle 00|\exp(a_1a_2\tanh r)a_1{}^q a_2{}^p|\eta,\tau_1,\tau_2\rangle. \tag{46}$$

In order to obtain the tomogram amplitude, we first obtain

$$\begin{aligned} &\langle 00|\exp(a_1a_2\tanh r)a_1{}^q a_2{}^p|\eta,\tau_1,\tau_2\rangle \\ &= A\langle 00|\exp(a_1a_2\tanh r)a_1{}^q a_2{}^p\int\frac{d^2z_1d^2z_2}{\pi^2}|z_1z_2\rangle\langle z_1z_2| \\ &\quad\times\exp[B + Ca_1^\dagger + Da_2^\dagger + Ea_1^\dagger a_2^\dagger - Fa_1^{\dagger 2} - Fa_2^{\dagger 2}]|00\rangle \\ &= A\int\frac{d^2z_1d^2z_2}{\pi^2}z_1{}^q z_2{}^p\exp[-|z_1|^2 - |z_2|^2 + z_1z_2\tanh r + B \\ &\quad + Cz_1^* + Dz_2^* + Ez_1^*z_2^* - Fz_1^{*2} - Fz_2^{*2}] \\ &= A\frac{\partial^{q+p}}{\partial\lambda^q\partial\xi^p}\int\frac{d^2z_1d^2z_2}{\pi^2}\exp(-|z_1|^2 - |z_2|^2 + z_1z_2\tanh r + B \\ &\quad + \lambda z_1 + \xi z_2 + Cz_1^* + Dz_2^* + Ez_1^*z_2^* - Fz_1^{*2} - Fz_2^{*2}). \end{aligned} \tag{47}$$

Carrying out the integral over d^2z_1 and d^2z_2 for (47) leads to

$$\begin{aligned} \text{Equation (47)} &= \frac{A}{\sqrt{M}}\frac{\partial^{q+p}}{\partial\lambda^q\partial\xi^p}\exp(-K\lambda^2 + (G\xi + T)\lambda - N\xi^2 + L\xi + P) \\ &= \frac{A}{\sqrt{M}}K^{q/2}e^P\frac{\partial^p}{\partial\xi^p}H_q\left(\frac{G\xi+T}{2\sqrt{K}}\right)\exp(-N\xi^2 + L\xi) \\ &= \frac{A}{\sqrt{M}}K^{q/2}e^P\sum_{i=0}^p\binom{p}{i}\left[\frac{\partial^i}{\partial\xi^i}H_q\left(\frac{G\xi+T}{2\sqrt{K}}\right)\right] \end{aligned}$$

$$\times \left[\frac{\partial^{p-i}}{\partial \xi^{p-i}} \exp(-N\xi^2 + L\xi) \right]$$

$$= \frac{A}{\sqrt{M}} K^{q/2} e^P \sum_{i=0}^{p} \binom{p}{i} \frac{2^i q!}{(q-i)!} \left(\frac{G}{2\sqrt{K}} \right)^i H_{q-i}\left(\frac{T}{2\sqrt{K}} \right)$$

$$\times N^{p-i} H_{p-i}\left(\frac{L}{2\sqrt{N}} \right), \tag{48}$$

where we have used the generating function formula of $H_n(x)$ [5]

$$H_n(x) = \frac{\partial^n}{\partial t^n} e^{2xt-t^2}|_{t=0} \tag{49}$$

and the recurrence relation of $H_n(x)$

$$\frac{d^l}{dx^l} H_n(x) = \frac{2^l n!}{(n-l)!} H_{n-l}(x) \tag{50}$$

and the parameters are given by, respectively

$$K = \frac{(2E + 4F^2 \tanh r - E^2 \tanh r) F \tanh r}{M} + F, \tag{51}$$

$$G = \frac{E + (4F^2 - E^2)\tanh r}{M}, \qquad N = \frac{F}{M}, \tag{52}$$

$$T = \frac{(CE - 2DF)\tanh r + (2F^2 - E^2) C \tanh^2 r}{M} + C, \tag{53}$$

$$L = \frac{D - (2CF + DE)\tanh r}{M}, \qquad P = \frac{(C - DF\tanh r) D \tanh r}{M} + B, \tag{54}$$

$$M = [(E + 2F)\tanh r - 1][(E - 2F)\tanh r - 1]. \tag{55}$$

So the tomogram amplitude of the state $|\phi\rangle$ is

$$\langle \phi | \eta, \tau_1, \tau_2 \rangle = \frac{C_{q,p}(r)(\cosh r)^{p+q+1}}{(\tanh r)^{(p+q)/2}} \frac{A}{\sqrt{M}} K^{q/2} e^P \sum_{i=0}^{p} \binom{p}{i} \frac{2^i q!}{(q-i)!}$$

$$\times \left(\frac{G}{2\sqrt{K}} \right)^i H_{q-i}\left(\frac{T}{2\sqrt{K}} \right) N^{p-i} H_{p-i}\left(\frac{L}{2\sqrt{N}} \right). \tag{56}$$

Using (43) we can obtain the tomogram of the state $|\phi\rangle$

$$T(\eta, \tau_1, \tau_2) = \left| \frac{C_{q,p}(r)(\cosh r)^{p+q+1}}{(\tanh r)^{(p+q)/2}} \frac{A}{\sqrt{M}} K^{q/2} e^P \sum_{i=0}^{p} \binom{p}{i} \frac{2^i q!}{(q-i)!} \right.$$

$$\left. \times \left(\frac{G}{2\sqrt{K}} \right)^i H_{q-i}\left(\frac{T}{2\sqrt{K}} \right) N^{p-i} H_{p-i}\left(\frac{L}{2\sqrt{N}} \right) \right|^2. \tag{57}$$

Therefore, experimentally one can measure the module-square of the wave function $|\phi\rangle$ in the entangled state $|\eta, \tau_1, \tau_2\rangle$ representation, then the tomogram of the state $|\phi\rangle$ is obtained and also the state $|\phi\rangle$ is measured.

6 Conclusions

In summary, we have used the entangled state representation of Wigner operator to derive the Wigner function of the THPS in an entangled way in the $\rho - \gamma$ phase space. The result may be useful for experimentalists because they usually regard the entanglement of the two-mode correlated quantum states as a physical resource in quantum communication. The tomogram of the THPS is calculated with the aid of newly introduced entangled state representation $\langle \eta, \tau_1, \tau_2|$ in quantum optics. We emphasize that tomogram of any two-mode correlated quantum state $|\phi\rangle$ is just $|\langle \eta, \tau_1, \tau_2|\phi\rangle|^2$.

References

1. Dodonov, V.V., Man'ko, V.I.: Theory of Nonclassical States of Light. Taylor & Francis, London (2003)
2. Wigner, E.: Phys. Rev. **40**, 749 (1932)
3. Agawal, G.S., Wolf, E.: Phys. Rev. D **2**, 2161 (1972)
4. Bergou, J.A., Hillery, M., Yu, D.Q.: Phys. Rev. A **43**, 515 (1991)
5. Wang, Z.X., Guo, D.R.: General Theory of Special Functions. Science Publishing, Beijingin (1965) (in Chinese)
6. Fan, H.Y., Ye, X.: Phys. Lett. A **175**, 387 (1993)
7. Fan, H.Y., Yu, G.C.: Mod. Phys. Lett. A **15**, 499 (2000)
8. Fan, H.Y., Ye, X.: Phys. Lett. A **51**, 3343 (1995)
9. Fan, H.Y., Zaidi, H.R., Klauder, J.R.: Phys. Rev. D **35**, 1831 (1987)
10. Fan, H.Y.: Entangled State Representations in Quantum Mechanics and Their Applications. Shanghai Jiao Tong University Press, Shanghai (2001) (in Chinese)